The initial motion of a gas bubble formed in an inviscid liquid

Part 1. The two-dimensional bubble

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The paper deals with the initial motion of a two-dimensional bubble starting from rest in the form of a cylinder with its axis horizontal. The theory is based on the assumptions of irrotational motion in the liquid round the bubble, constant pressure within the bubble, and small displacements from the cylindrical form. This theory predicts that the bubble should rise with the acceleration of gravity, over a distance of at least the initial bubble radius, and that a tongue of liquid should be projected up from the base of the bubble into its interior. These predictions are confirmed by experiments which also show how the vorticity necessary for steady motion in the spherical-cap form is generated by the detachment of two small bubbles from the back of the main bubble.

1. Introduction

When a bubble is suddenly formed in a liquid, it tends to rise, and simultaneously to distort, under the influence of buoyancy forces. This paper deals with the upward motion of a two-dimensional bubble which starts from rest with a circular shape and then is free to rise under gravity forces. The subsequent distortions when such a bubble is suddenly formed in water have been examined photographically, and the results compared with theory for a bubble in an inviscid liquid of zero surface tension.

The work was begun because of interest in the formation of bubbles at an orifice through which air is blown steadily into a liquid of small viscosity. Work on this problem was described in a previous paper (Davidson & Schüler 1960), in which the theory was based on the assumption that the bubbles, as they form, are spherical. Experimental observations showed that this assumption is incorrect, since although each bubble forming at the orifice begins by being nearly spherical, the upward motion due to buoyancy soon produces distortion into a shape which approaches the spherical-cap form described by Davies & Taylor (1949) for a bubble in steady motion.

As a first step towards explaining these distortions, a study was made of the upward motion of a two-dimensional bubble which began as a cylindrical cavity filled with air. The theory to explain the subsequent motion was based on the assumption of irrotational motion in the liquid, and by adding together a series of harmonics it was possible to obtain an expression for the velocity potential to give constant pressure at the surface of the bubble, surface tension being neglected. This theory, valid only for small displacements from the cylindrical form, predicts that the bubble should have an initial upward acceleration of g, and that the bubble should distort into the form of the cross-section of the head of a mushroom, a tongue of liquid being projected upwards from the bottom of the bubble. These predictions are confirmed by experiment; the results also show how the flow changes from the irrotational motion to the fully separated flow described by Davies & Taylor (1949).

The foregoing theory has been adapted to the more complex case of a growing spherical bubble formed in an inviscid liquid. The results from this theory are in reasonable agreement with the measurements of Davidson & Schüler (1960) and will be published in Part 2 of the present paper.

It is believed that a theory of the kind outlined above would be applicable to the study of nucleate boiling, the initial motion of a bubble produced by a submarine explosion, and the initial motion of the fireball generated by an atomic explosion. The theory given in this paper is related to that of Hartunian & Sears (1957), who, however, were concerned with small bubbles in which surface tension is important; consequently they used rather different methods. The analysis given here also resembles that of Penney & Price (1950), although they were concerned only with bubble pulsations when there was no translation of the bubble.

2. The theory of the initial motion

To calculate the motion, induced by gravity, of an initially cylindrical bubble in an inviscid liquid, we express the velocity potential ϕ in terms of a series of harmonics whose coefficients have to be adjusted to give constant pressure within the bubble. The series is

$$\phi = \frac{Ua^2\cos\theta}{r} + \sum_{n=2}^{\infty}\frac{\beta_n}{r^n}\cos n\theta. \tag{1}$$

Here a is the initial radius of the bubble, r and θ are polar co-ordinates whose origin moves with the bubble and has an upward velocity U at time t after the start. The coefficients β_n are functions of time, and have to be adjusted so that the pressure within the bubble shall be independent of θ. The pressure p at any point in the liquid is found from Bernoulli's theorem

$$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - \tfrac{1}{2}q^2 + K - g\left(\int_0^t U\,dt' + r\cos\theta\right), \tag{2}$$

where $\partial\phi/\partial t$ is the partial derivative at a point fixed in space, q is the absolute velocity of the liquid, and K is constant since the pressure at infinity is presumed fixed. Following the method described by Lamb (1932), $\partial\phi/\partial t$ is calculated in terms of $\dot{r} = -U\cos\theta$ and $\dot{\theta} = U\sin\theta/r$, the rates of change of r and θ for a point fixed in space, so that from (1)

$$\frac{\partial\phi}{\partial t} = \sum_{n=1}^{\infty}\left[\frac{\dot{\beta}_n}{r^n}\cos n\theta + \frac{n\beta_n U}{r^{n+1}}\cos(n+1)\theta\right], \tag{3}$$

where $\beta_1 = Ua^2$. Also, $q^2 = (\partial\phi/\partial r)^2 + (\partial\phi/\partial\theta)^2/r^2$, and again using (1),

$$q^2 = \frac{U^2a^4}{r^4} + \sum_{n=2}^{\infty}\left[\frac{2nUa^2\beta_n}{r^{n+3}}\cos(n-1)\theta + \frac{n^2\beta_n^2}{r^{2n+2}} + 2\sum_{m=2}^{n-1}\frac{nm\beta_n\beta_m}{r^{n+m+2}}\cos(n-m)\theta\right]. \tag{4}$$

To calculate the pressure p_s just outside the bubble, (3) and (4) are substituted into (2) with $r = R$, where R is the radius vector from the moving origin of (r, θ) to the surface of the bubble. Then R is written as $a(1+\zeta)$, where ζ, a dimensionless function of t, θ and a, describes the shape of the bubble, and it is assumed that $\zeta \ll 1$, so that

$$\begin{aligned}\frac{p_s}{\rho} = K - g\int_0^t U\,dt' - \tfrac{1}{2}U^2 + \left[\frac{1}{a}\frac{d}{dt}(Ua^2) - ga\right]\cos\theta + U^2\cos 2\theta \\ + \sum_{n=2}^{\infty}\left\{\frac{\dot\beta_n}{a^n}\cos n\theta + \frac{nU\beta_n}{a^{n+1}}[\cos(n+1)\theta - \cos(n-1)\theta] - \frac{n^2\beta_n^2}{2a^{2n+2}}\right. \\ \left. - \sum_{m=2}^{n-1}\frac{nm\beta_n\beta_m\cos(n-m)\theta}{a^{n+m+2}}\right\} + O(\zeta).\end{aligned} \tag{5}$$

Experimental observations show that the volume of a rising bubble does not change much, and it will therefore be assumed that a is constant, so that the bubble is imagined to be filled with incompressible fluid of zero density. Also, surface tension is neglected, and consequently p_s must be equal to the pressure within the bubble, and must be independent of θ. Therefore in (5) the coefficients of $\cos n\theta$ must be zero, leading to the following series of equations:

$$n = 1, \quad \frac{d}{dt}(Ua^2) - ga^2 - \sum_{m=1}^{\infty}\frac{m(m+1)\beta_m\beta_{m+1}}{a^{2m+2}} = 0, \tag{6}$$

$$n \geqslant 2, \quad \dot\beta_n + (n-1)U\beta_{n-1} - \sum_{m=1}^{\infty}\frac{m(m+n)\beta_m\beta_{m+n}}{a^{2m+2}} = 0, \tag{7}$$

where $\beta_1 = Ua^2$ as before. In order to render these equations tractable, we must ignore the non-linear terms under the summation signs, and the first approximation is thus:

$$\left.\begin{aligned}&\text{from (6),} && \beta_1^{(1)}/a^2 = U^{(1)} = gt,\\ &\text{from (7) for } n \geqslant 2, && \beta_n^{(1)} = \frac{(-1)^{n-1}(n-1)!\,n!\,2^na^2}{(2n)!\,t}(gt^2)^n,\end{aligned}\right\} \tag{8}$$

the latter being obtained by repeated integration $n-1$ times. We may now insert these values of $\beta_n^{(1)}$ in the first of the non-linear terms in (6) and (7) and integrate again to obtain, as a second approximation,

$$\left.\begin{aligned}\beta_1^{(2)}/a^2 = U^{(2)} = gt(1 - \tfrac{2}{15}N^2),\\ n \geqslant 2, \quad \beta_n^{(2)} = \beta_n^{(1)}\left[1 - \frac{n(n+1)N^2}{(2n+1)(2n+3)}\right],\end{aligned}\right\} \tag{9}$$

where $N = gt^2/a$. In the Appendix it is shown that the equations (9) are accurate to about 8 % for $N < 1$. Integration of the first of equations (9) gives the distance s that the centre of co-ordinates has risen in time t,

$$s/a = \tfrac{1}{2}N(1 - \tfrac{2}{45}N^2). \tag{10}$$

To calculate the shape of the bubble, the normal velocity at its surface $a\dot{\zeta}+U\cos\theta$ is equated to $-(\partial\phi/\partial r)_{r=R}$, giving

$$\dot{\zeta} = \sum_{n=2}^{\infty} \frac{n\beta_n}{a^{n+2}} \cos n\theta - \zeta \sum_{n=1}^{\infty} \frac{n(n+1)\beta_n}{a^{n+2}} \cos n\theta, \tag{11}$$

the $O(\zeta)$ term having been retained in this equation. It is not easy to integrate (11), but since the expressions under the summation signs are odd polynomials in t, and converge like a geometrical progression, it is reasonable to try a solution of the form

$$\zeta = A_1 N^2 + A_2 N^3 + A_3 N^4 + \ldots. \tag{12}$$

Substituting from (9) and (12) into (11) and equating coefficients of powers of t then gives for the distortion ζ, using only the first four terms,

$$\begin{aligned}\zeta = &-0\cdot167N^2\cos 2\theta + 0\cdot0944N^3(\cos 3\theta + 0\cdot294\cos\theta)\\ &-0\cdot0162N^4(\cos 4\theta + 0\cdot0162\cos 2\theta + 0\cdot397) + 0\cdot0416N^5\\ &\times(\cos 5\theta - 0\cdot0203\cos 3\theta + 0\cdot680\cos\theta).\end{aligned} \tag{13}$$

Figure 3 shows bubble shapes calculated from this equation, and figure 5 the vertical diameter D of the bubble as a function of N.

The theoretical movement of the centroid is a convenient quantity to compare with experimental results, since it is representative of the movement of the whole bubble, and it can be obtained by adding the coefficient of $\cos\theta$ in (13) to the movement s of the centre of co-ordinate axes from (10). This gives a displacement

$$s_g = 0\cdot5Na + 0\cdot0056N^3a, \tag{14}$$

neglecting higher powers of N. We now consider the radial deviation ξ of the bubble from a circle of radius a centred on a point G whose upward displacement is given by (14). ξ will be given by a series like (13) but containing no $\cos\theta$ terms, and therefore

$$\int_0^{\pi} \xi \cos\theta \, d\theta = 0. \tag{15}$$

Now provided ξ is small compared with a, (15) is the condition that G shall be the centroid of the bubble. It will be seen that (14) gives a movement almost the same as that due to a constant acceleration of g.

3. Experiments with a two-dimensional bubble

For comparison with the above theory, an air bubble was formed in water between two vertical Perspex plates $\frac{3}{8}$ in. apart, as shown in figure 1. The column of water was 10 in. wide and 4 ft. high, and the air bubble was formed by suddenly withdrawing the brass tube shown in the figure. The impulse I to withdraw the tube was applied by a steel cable passing over a pulley and connected through a spring to a horizontal plate on to which a weight was dropped from a height of 4 ft. In this way an effectively two-dimensional bubble was formed in the water within 0·005 sec. The air pressure within the tube was such that there was minimal tendency for a volume change to occur on release.

The subsequent motion of the bubble was photographed either by a Fastax camera at 2000 frames/sec or by a Pathé camera at 80 frames/sec, and a typical series of pictures is shown in figure 2, plate 1. With the Fastax camera, the zero time was marked by a neon bulb viewed by the camera and lit when a micro-switch was tripped by withdrawal of the tube.

The film was analysed frame by frame by projecting it and drawing the shape of the enlarged image of the bubble from each frame on to a sheet of cardboard, the original centre of the bubble also being marked as a reference point. The shape was then cut out and each model was suspended from two points to locate

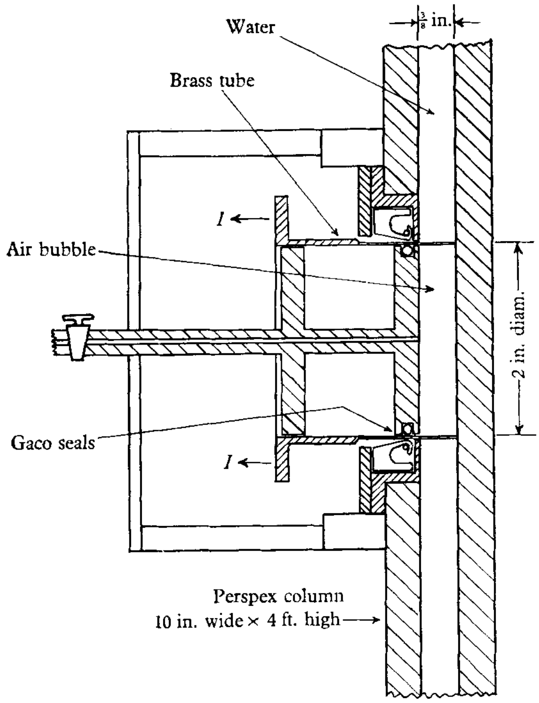

FIGURE 1. Apparatus for studying two-dimensional bubbles.

the centroid, and the distance s_g that it had risen was measured in the early stages with a travelling microscope and subsequently with a centimetre rule. The vertical diameter of the bubble was also measured and the results are shown in figures 4 and 5.

4. Comparison between theory and experiment

Figure 3 shows a direct comparison of bubble shapes, as measured experimentally and as calculated theoretically from (13). The centroids are shown, and each theoretical shape is placed so that its centroid coincides with the centroid of the observed bubble. The two sets of shapes show general agreement, and it is satisfactory that the theory predicts the formation of the tongue of liquid that comes up from the bottom of the bubble. This tongue is characteristic of bubbles forming at an orifice, and good photographs of it are given by Helsby & Tuson (1955). Hitherto it has not been clear whether the tongue is due to the following bubble; it is now certain that the tongue is characteristic of an accelerating isolated bubble.

Figure 4 shows the measured upward movement compared with the theoretical result (14). The zero as marked is the time at which the brass tube had just begun to move. The time of complete withdrawal of the tube was 0·005 sec, which is equivalent to a zero error of 0·098 on the abscissa and the thin line on figure 4

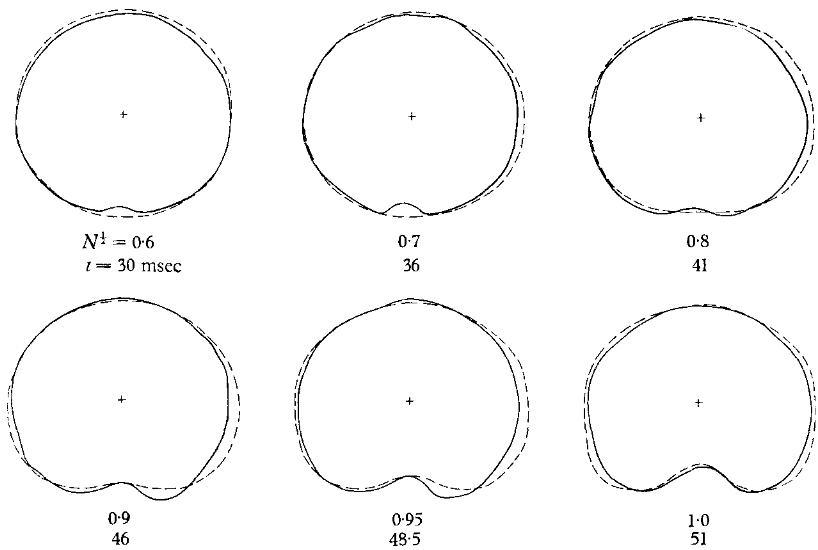

FIGURE 3. Experimental and theoretical bubble shapes compared: ——, experimental; ----, theoretical.

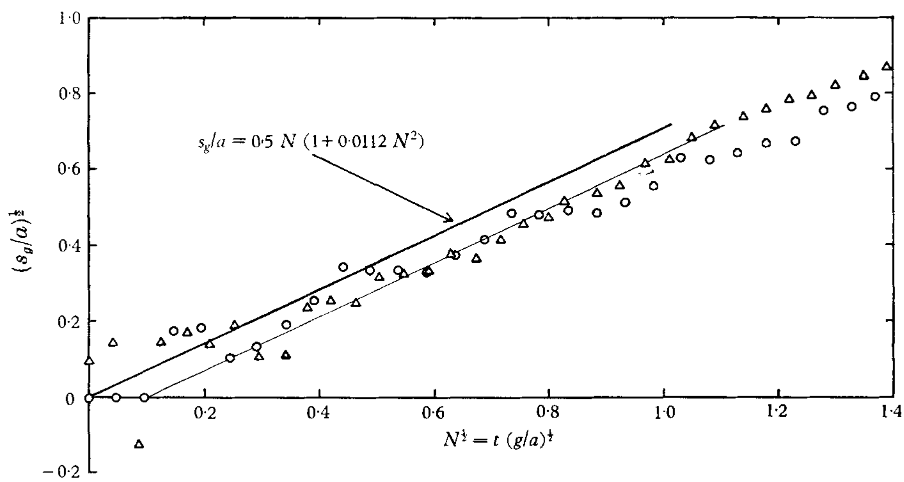

FIGURE 4. The upward displacement of a 2 in. dia. bubble starting from rest: △, bubble 1; ○, bubble 2.

is obtained from equation (14) with this point as origin. The results are seen to be in good agreement with the shifted line, and it would thus seem that the bubble does not begin to move until the tube has been completely withdrawn from the column. There is some scatter at the lower end of the curve, but the errors in measuring the displacements there are comparable with the displacements themselves so their significance is not so great as would appear at first sight.

The results shown in figure 4 confirm the theoretical prediction from (14), that the acceleration is almost exactly g up to $N = 1$. Equation (14) is effectively valid only for N less than unity, because of the omission of higher powers of N, but the experimental results show that the uniform upward acceleration of g is maintained when N is greater than unity. This confirms to some extent the assumption of Davidson & Schüler (1960), which was that an initially spherical

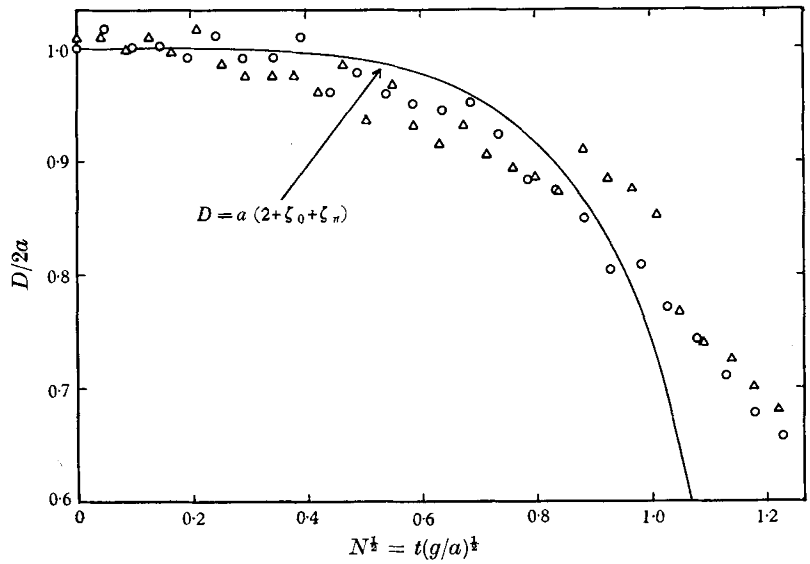

FIGURE 5. The change in vertical diameter of a 2 in. dia. bubble starting from rest: △, bubble 1; ○, bubble 2.

bubble starting from rest would have an upward acceleration of $2g$; this is equivalent to an acceleration of g in the two-dimensional case, and will be further considered in Part 2.

Figure 5 shows the vertical diameter of the bubble, as a measure of its distortion from the cylindrical form, compared with the theory obtained from (13). The measured shapes in figure 3 show that considerable scatter is to be expected but it is difficult to see why the distortion should be greater than the theoretical for $N < 1$; for the effects of surface tension and viscosity should make the distortion less. The zero error mentioned above would make these differences more marked, but it is probable that the presence of the Perspex column walls, the limited extent of the fluid, and the slight disturbance caused by the withdrawal of the tube would account for the differences. It should be pointed out that the theory is valid only for $N < 1$.

A final point that emerges from a study of the photographs shown in figure 2, plate 1, is the way in which the flow round the bubble changes from the irrotational motion of the above theory to the fully separated flow necessary for steady motion of the bubble in the spherical-cap form described by Davies & Taylor (1949). The change takes place by the detachment of two small bubbles from the main bubble at its lower extremities, each of the two detached bubbles being at the centre of a vortex. This mode of generating vorticity behind the main bubble by the detachment of two smaller bubbles may be contrasted with the way in which vorticity is generated in the wake of a solid object during its acceleration from rest. In the latter case the growth and detachment of boundary layers plays an essential part, whereas the bubble succeeds in generating vorticity —by breaking up—without any help whatever from viscous forces.

Appendix

Non-linear terms in (6) *and* (7)

Using the first approximation $\beta_n^{(1)}$, the series in (7) becomes

$$\sum_{m=1}^{\infty} \frac{(-1)^n [m!\,(m+n)!]^2\, 2^{2m+n} (gt^2)^{2m+n}}{(2m)!\,(2m+2n)!\, a^{2m-2} t^2},$$

and the ratio of the mth to the $(m-1)$th term, after integration with respect to t, is

$$\frac{m(m+n)\,(4m+2n-5)\,N^2}{(2m-1)\,(2m+2n-1)\,(4m+2n-1)}.$$

This tends to $N^2/4$ when m is large, and hence by comparison with a geometrical progression the sum of the above series is close to $1/(1-N^2/4)$ times the first term, provided $N<2$. With $N=1$, the error introduced into (7) by neglecting all terms but the first in the infinite series can be shown to be close to one-third of the first term; now from (9) the first term is seen to be $n(n+1)/(2n+1)(2n+3)$ times $\beta_n^{(1)}$, the first approximation. Hence for large n, the total error introduced by neglecting the second and higher terms in (7) is close to $\beta_n^{(1)}/(3\times 4)$, i.e. 8 % of the first approximation.

$O(\zeta)$ term in (5)

A referee has pointed out that the $O(\zeta)$ terms in (5) give rise to a term of order $(gt^2)^{n+1}$ in (7), and that this affects the second approximation for β_n. Closer inspection of equations (3), (4) and (5) shows that in (5)

$$\begin{aligned}\frac{O(\zeta)}{\zeta} = &-\left[\frac{\ddot{\beta}_1}{a}+ga\right]\cos\theta + 2U^2(1-\cos 2\theta)\\ &+\sum_{n=2}^{\infty}\left\{-\frac{\ddot{\beta}_n}{a^n}\,n\cos n\theta - \frac{nU\beta_n}{a^{n+1}}\left[(n+1)\cos(n+1)\theta-(n+3)\cos(n-1)\theta\right]\right.\\ &\left.+\frac{(n+1)\,n^2\beta_n^2}{a^{2n+2}}+O(\beta_n\beta_m)\right\}.\end{aligned}$$

Substituting for ζ from (13), using only the first term, we see that the coefficient of $\cos\theta$ in the expression for $O(\zeta)$ is

$$0{\cdot}0835N^2(\ddot{\beta}_1 a^{-1}+ga)+O(t^8),$$

and insertion of the first approximation for $\ddot{\beta}_1$ $(=ga^2)$ then simplifies this coefficient to $0{\cdot}167N^2ga$, leaving out the $O(t^8)$ term. The second approximation for U from (6) should thus be obtained by integrating

$$\frac{d}{dt}(U^{(2)}a^2) - ga^2 - \frac{2\beta_1^{(1)}\beta_2^{(1)}}{a^4} + 0{\cdot}167N^2ga^2 = 0,$$

which leads to $U^{(2)} = gt(1-\frac{1}{6}N^2)$, which can itself be integrated to give

$$s/a = \tfrac{1}{2}N(1-N^2/18). \tag{16}$$

The last two equations differ from (9) and (10) by rather less than the 8 % inaccuracy already described in the first part of this Appendix; all these errors mean that the second term in each of the square brackets of (9) is of rather doubtful value. However, it is easily shown that these terms affect only N^4 and higher powers in (13).

Finally, it is worth noting that the use of (16) in place of (10) leads to (14) becoming $s_g = 0{\cdot}5Na$, so that the bubble has an acceleration of exactly g throughout the period when these approximations are valid.

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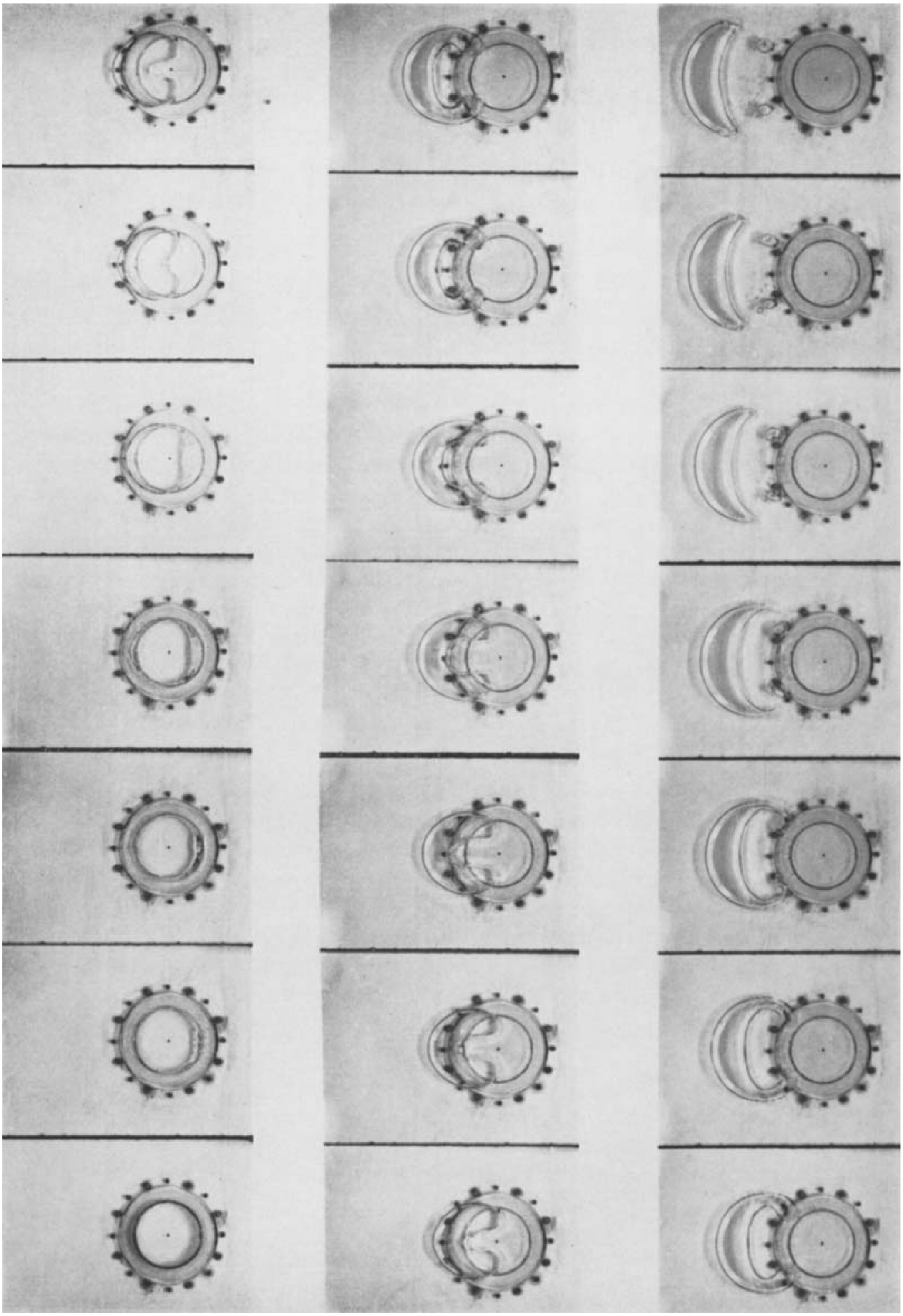

FIGURE 2. Ciné pictures, at about 80 frames/sec, of the initial motion of a 2 in.-dia. bubble.